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STUDENTS' INVESTIGATION OF A VIEW TUBE

Using multiple representations and technology, students explore the relationship between tube length and field of vision.

The inquiry-based approach to learning has gained popularity in recent times. Those who promote this approach maintain that mathematical investigation should be used to engage students (Baroody and Coslick 1998). Diezmann, Watters, and English (2001) note that "mathematical investigations are contextualized problem solving tasks through which students can speculate, test ideas and argue with others to defend their solutions" (p. 170). The National Council of Teachers of Mathematics (NCTM 2000) recommends that problem solving be the center of mathematics teaching in promoting student learning through problem-solving contexts and inquiry-based environment. Inquiry means involvement that leads to understanding and exposes students to mathematical investigation. Investigation exposes students to different forms of representation. NCTM states that "the term *representation* refers both to process and to product—in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself" (NCTM 2000, p. 67).

There are compelling reasons for using investigation in the instructional process with students of different abilities and at all grade levels.

First, mathematical investigation provides students the opportunity to apply what they have learned in mathematics to new situations. Investigation allows them to model mathematical situations in real-life applications (NCTM 2000).

Second, investigation requires time and planning. Such investigations with single or multiple solutions that can be reached by making conjectures may strengthen students' problem-solving skills and help them realize that there is often more than one way to solve a mathematical problem (Greenes 1996; Kissane 1988).

Third, investigation can promote cooperative learning (Andrini 1991). Investigations are usually done in groups, in which students communicate among themselves as they share ideas, conjecture, and test these conjectures. Cooperative learning promotes student learning, oral communication, and social skills (Johnson and Johnson 1990).

Fourth, investigation requires that students carry out the tasks that enable them to learn by doing, diminishing the notion that mathematics is a difficult subject.

THE INVESTIGATION

Three related investigations were presented to a class of secondary school preservice teachers:

1. Investigate the relationship between the length of the view tube (x) and the viewable vertical distance on the wall (y) while maintaining a perpendicular line from the eye to the wall and keeping the diameter of the view tube constant. Use view tubes of different lengths for comparison.
2. Investigate the relationship between the perpendicular distance from the eye to the wall (x) and the viewable vertical distance on the wall (y) using a view tube of constant length and diameter.
3. Investigate the relationship between the diameter of the view tube (x) of constant lengths and the viewable vertical distance on the wall (y) while keeping the perpendicular distance from the eye to the wall constant. Use view tubes of different diameters for comparison.

Fig. 1 Students measure the length of the view tube (x) and the viewable vertical distance on the wall (y).

This article reports on only the third investigation—the relationship between the length of the view tube (x) and the viewable vertical distance on the wall (y) while keeping the perpendicular distance from the eye to the wall and the diameter of the view tube constant. (This activity has been suggested by Day et al. [2001] and Wilson and Shealy [1995].)

DATA COLLECTION AND TABULAR REPRESENTATION

The class of preservice teachers was divided into three groups of three members each. Before the investigation, each group was provided with a meterstick, TI-84 calculators, graph paper, and eleven PVC view tubes of different lengths. (Easily available materials such as inner tubes from rolls of paper towels, bathroom paper, or wrapping paper can be used for the investigation. PVC tubes of different lengths and diameter are available at many construction sites.) Each of the three students in the group had a different task: One was to look through the view tube, one was to measure what the first student was seeing through the tube, and the third was to record the measures. The students switched roles so that each could look through the view tube. Before the investigation the students were asked to predict what the relationship might be—linear, power, or exponential.

The students started the activity by arranging the eleven view tubes from shortest to longest. When asked why they began with the smallest view tube and proceeded by order of size, the students gave several reasons:

- "We arranged them that way because it was easier for us to keep track of what view tubes we used to avoid leaving any out."
- "We arranged them that way because it is kind of natural to proceed from the small one to the big

Table 1 Length of the View Tube (x) and Viewable Vertical Distance (y) (in.)

x	y
1	129.5
3	51.25
4.5	37.5
6	31.5
7	27.5
10	19
12	15.5
14	13.5
16	12
19	10.75
27.25	7.5

one. In algebra, when plotting algebraic equations, we were taught to start with 0 and then 1, 2, …., therefore it makes sense to do the same here."

When asked what would happen if the tubes were used in random order, the students simply said that this procedure was not conventional. Also, although the students did not mention this reason explicitly, using the tubes in order of increasing size makes it easier to understand the relationship between the variables in the problem.

Students picked a spot on the floor (153 in. from the wall) and marked it as a viewing point. They started with the shortest view tube and measured its length (x_1), and every student in the group used the tube to measure the vertical distance he or she could see on the wall (see **fig. 1**). They then averaged the observations made for the tube. This procedure was repeated for all tubes. The resulting data for one group are shown in **table 1**.

GRAPHICAL REPRESENTATION

Students were asked to analyze the data they generated and make conjectures concerning the relationship between the variables. They compared the x-values and y-values to see whether a relationship existed. They noted that each x-value was approximately 2 units greater than its predecessor and tried to find a formula for generating the x-values but made little progress. Moving to the y-values, they attempted to discern any relationship there. They tried to find a mathematical formula for generating the y-values but, again, had little success.

The students then decided to look at ordered pairs (x, y) simultaneously. Some students conjectured that as the x-value increases, the y-value decreases and suggested that the relationship might be "hyperbolic." Those students who made conjectures were able to do so because they had arranged the view tubes from the shortest to the longest.

To make sense of the relationships that they suspected, students used graph paper to plot their data points, which they connected with a smooth curve (see **fig. 2**). At this point, every student in the group agreed that the relationship was "hyperbolic," in agreement with earlier conjectures made by some of the group members. Some students were surprised to find that the relationship was not "inverse" because as the tube length (x) becomes smaller, what is visible (y) grows rapidly.

Students were able to interpolate and extrapolate data from their graphs. For instance, when using a view tube of length 13 in., they found that they could see a vertical distance of about 14.5 in. on the wall. Likewise, they concluded that in order to see a vertical distance of 80 in. on the wall, they would need a view tube of length 1.5 in.

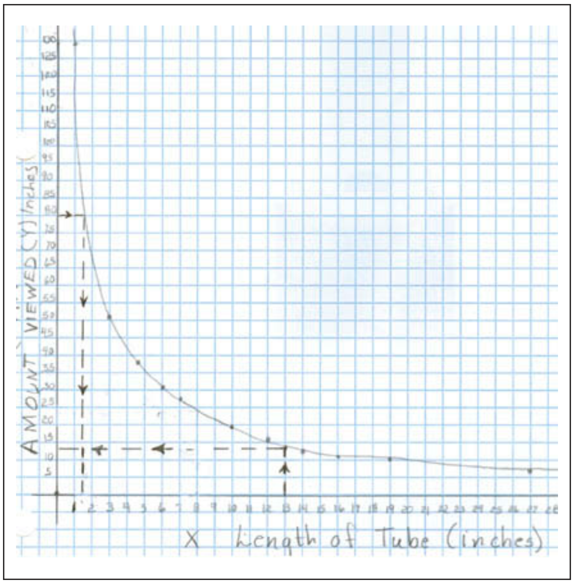


Fig. 2 Connecting the data points with a smooth curve led students to think of a "hyperbolic" model for the data.

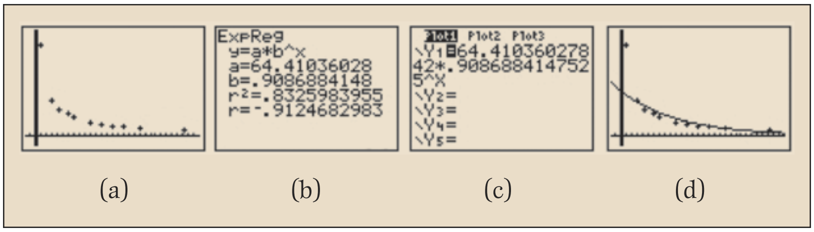


Fig. 3 The data (**a**), regression equation and correlation coefficient (**b**), y = menu (**c**), and curve superimposed on data (**d**) for exponential regression made students look further.

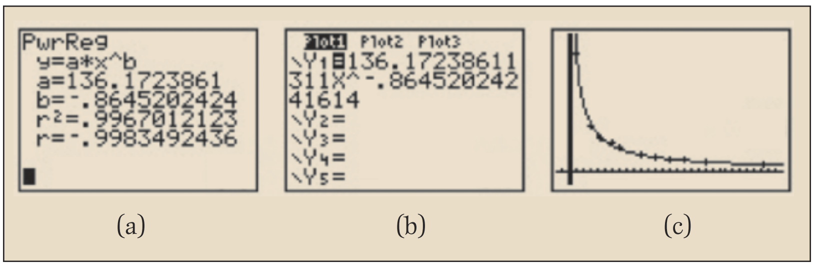


Fig. 4 The regression coefficient (**a**), y = menu (**b**), and curve superimposed on data (**c**) for power regression gave better results.

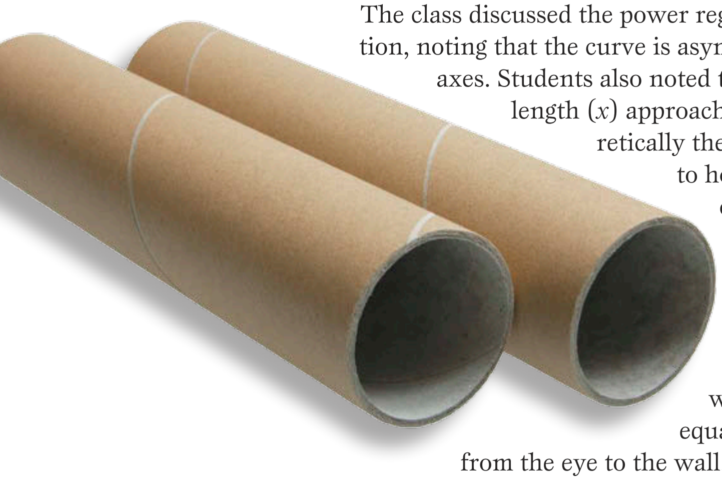
ALGEBRAIC REPRESENTATION

Students were interested in investigating the equation of the curve of best fit and discussed what was the best method to use to find it. One suggestion was to use the TI-84 graphing calculator to fit a mathematical model to the data. Students were not sure what kind of model would best fit the data, so they tried two models: exponential and power. The output produced using exponential regression ($y = ab^x$) led to the equation $y = 64.4\ (0.908)^x$, with a coefficient of determination $r^2 = 0.832$ (see **fig. 3**). The coefficient of determination of 0.832 led students to try power regression.

Power regression led to the equation $y = 136.17x^{-0.864}$ (see **fig. 4**). On the basis of the coef-

ficient of determination ($r^2 = 0.996$), students decided to use the power model as a curve of best fit (see **fig. 4c**).

The class discussed the power regression equation, noting that the curve is asymptotic to both axes. Students also noted that as the tube length (x) approaches zero, theoretically there is no limit to how far the eye can see. On the other hand, x must be less than the distance from the eye to the wall. When x equals the distance from the eye to the wall, we know that y should equal the diameter of the view tube and, therefore, that y will never be zero. The students also noted that the equation can be used to interpolate data and extrapolate other values. Many questions about regression equations, however, were not answered.



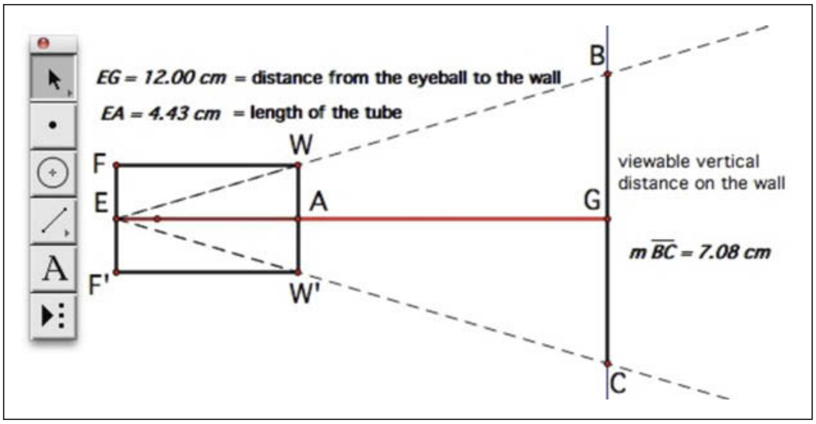


Fig. 5 Students were able to use GSP to model the length of the view tube and its viewable vertical distance on the wall but were not able to tabulate data to make meaningful conclusions.

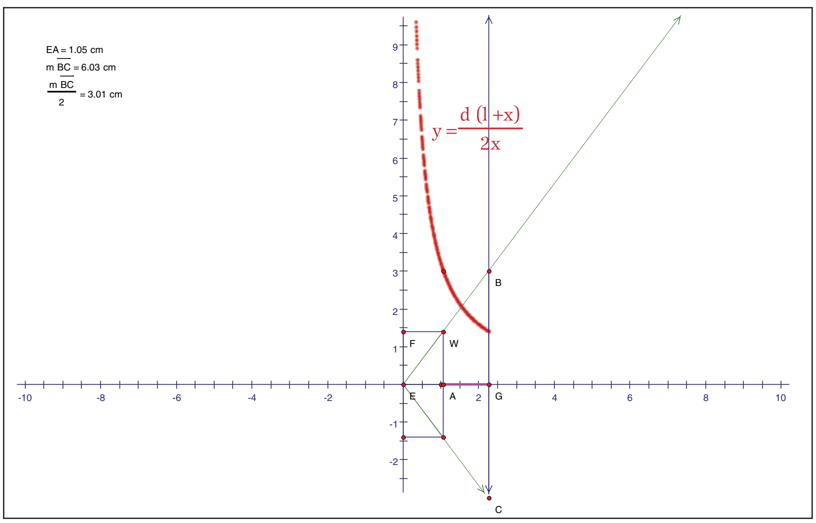


Fig. 6 In this diagram, x represents the length EA, l the distance from the end of the view tube to the wall, and d the diameter of the tube.

GEOMETRIC REPRESENTATION

Students then were asked to model the view tube investigation with the aid of The Geometer's Sketchpad™ (GSP) software. **Figure 5** shows the results of work by a group of four students.

Some students pointed out that they could easily generate data—the length of the tube (x) and the viewable vertical distance (y)—using the model but were not sure how to use GSP to represent the relationship in graphical form. One student suggested that the length of the tube (x) and viewable vertical distance (y) could be plotted using the graph menu in GSP. As a result, the whole class was involved in enriching the model (see the GSP instructions in the **appendix**).

DISCUSSION OF THE MODEL

In **figure 6**, the vertical distance BC (how much can be seen on the wall) is twice the dependent variable (y), while EA is the independent variable (x), the tube length. We use l, x, d, and $2y$ to represent, respectively, the distance from the end of the tube to the wall, length of the tube, diameter of the tube, and viewable vertical distance. In similar triangles EWW' and EBC,

$$\frac{EA}{EG} = \frac{WW'}{BC}.$$

Then, making the appropriate substitutions, we obtain the following:

$$\frac{x}{l+x} = \frac{d}{2y} \rightarrow 2xy = d(l+x)$$

$$y = \frac{d(l+x)}{2x} = \frac{dl}{2x} + \frac{d}{2}$$

The students were asked what the equation

$$y = \frac{d(l+x)}{2x} = \frac{dl}{2x} + \frac{d}{2}$$

meant in relation to the model. They noted that as the length of the tube (x) gets smaller, the visible amount grows very quickly. That is,

$$\lim_{x \to 0}\left(\frac{dl}{2x} + \frac{d}{2}\right) = +\infty.$$

Note that when $x = 0$, there is no tube and therefore what is visible is unlimited; this concept made sense to the students. As the length of the tube increases, what is visible decreases, which means that

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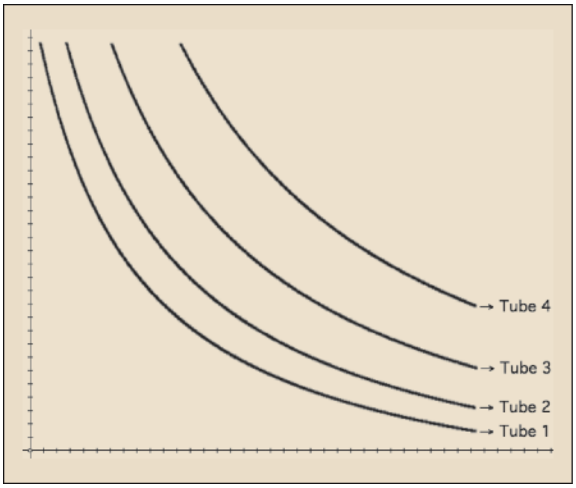

Fig. 7 At first, students had trouble matching curves to tube lengths.

$$\begin{aligned}\lim_{x\to(x+l)}\left(\frac{dl}{2x}+\frac{d}{2}\right) &= \lim_{x\to(x+l)}\left(\frac{d(l+x-x)}{2x}+\frac{d}{2}\right)\\ &= \lim_{x\to(x+l)}\left(\frac{d(l+x)}{2x}-\frac{dx}{2x}+\frac{d}{2}\right)\\ &= \frac{d}{2}-\frac{d}{2}+\frac{d}{2}=\frac{d}{2}.\end{aligned}$$

Therefore, as $x \to (x+l)$, $y \to d/2$. This analysis made sense to students because if the length of the tube is equal to the distance from the eye to the wall (the view tube is touching the wall), then they can see only the vertical distance equivalent to the diameter of the tube.

Students were then given curves produced by generating data from four tubes of different diameters and asked to discuss their dimensions (see **fig. 7**). This task was not an easy one. Some students referred to the GSP model (see **fig. 6**) to get ideas to enable them to generalize. Holding and dragging the GSP model at point W, the students noted that the curve shifted upward as the diameter was increased and shifted downward when the diameter was decreased. They also pointed out that the equation

$$y=\frac{dl}{2x}+\frac{d}{2}$$

supports the idea that as diameter d increases, the y-value increases. On this basis, students indicated that tube 4 had the biggest diameter, followed by tube 3, tube 2, and finally tube 1 (see **fig. 7**).

CONCLUSION

This activity is an example of mathematical modeling and how it can be used in the learning and

teaching of mathematics. In this simple investigation, secondary school preservice teachers used different types of problem-solving strategies and several types of technology. Incorporating the TI-84 graphing calculator enabled them to use their knowledge of statistics in solving the problem, while GSP served as a visual model.

Communication with others while working in groups was important to the preservice teachers in this investigation. Collaborative learning also helped those who had not participated in class before to make meaningful contributions.

Moreover, the investigation gave the teachers the opportunity to learn by doing. They formulated conjectures and tested them by using data that they collected. The activity provided powerful evidence of the value of learning by doing.

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Appendix: Instructions for The Geometer's Sketchpad (GSP)

1. Open a GSP Sketch. From the Graph menu, choose Grid Form and then Square Grid. Then hide the grids by choosing the option in the Graph menu.
2. Label the origin (0, 0) as E, and let G be another point on the x-axis.
3. Construct line j perpendicular to the x-axis and passing through G. Construct segment EG.
4. Locate a point on $\overline{EG}$, label it A, and construct line n perpendicular to $\overline{EG}$ through A. On n construct a point W.
5. Let m be a line perpendicular to n passing through W and intersecting the y-axis at F. Let k be the segment between E and A. Let l be the segment FW.
6. Double-click on the x-axis to select a mirror line. Then select point W, segment WF, and point F. In the Transformation menu, select Reflect to construct points F' and W' and segment $F'W'$.
7. Construct segments FF' and WW'. Construct rays from E through W and from E through W' to meet the perpendicular line j at B and C, respectively.
8. Construct segments EW and EW'. Construct segment BC.
9. Let $m\overline{BC}$ = the length of $\overline{BC}$, and let EA = the distance from point E to point A.
10. Select the measured distance EA and length $m\overline{BC}$ in that order. From the Graph menu, select Plot as (x, y) to plot the point $(EA, m\overline{BC})$ on the sketch.
11. Select lines m and n and hide them using the Display menu.
12. Select the point $(EA, m\overline{BC})$. In the Display menu, click on Trace Point and then deselect the point.
13. Select point A. From the Display menu, select Animate Point.

For an Illuminations applet illustrating this activity, go to http://illuminations.nctm.org/ActivityDetail.aspx?ID=41.

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